# Mathematical Techniques

*Topic 5 – Lecture 3: Introduction to Integral Calculus*

*Applying  Integral Calculus*

# Scope and Coverage

*This topic will cover:*

- Applications of integration to find the area under a curve

# Learning Outcomes

*By the end of this topic students will be able to:*

- Apply integral calculus to solve a range of mathematical problems relating to the area under a curve

# Applying the Area Under a Curve

- We have looked so far at calculating the area under a curve and the area bound between a straight line and a curve. Both of these can be calculated by applying integral calculus.
- Other applications of integration also include calculating
  - Distance Travelled
  - Velocity of a moving object
  - Acceleration of a moving object
- It is also possible to calculate the volume of complex shapes using integration

# Velocity, Acceleration & Distance

When we looked at differentiation we were able to determine that if an object travels s metres in a time of t seconds then the velocity (v) of the body can be expressed in terms of

$$V = \frac{ds}{dt}$$

This equation can however be expressed through integration to determine the distance travelled by an object in relation to time and velocity.

This equation is given as $s = \int v \, dt + c$

# Velocity and Acceleration

This approach can also be applied to determine the velocity of an object in respect of its acceleration.

If we consider that the acceleration of a body is expressed as

$$a = \frac{dv}{dt}$$

Then we can integrate to find the velocity by $v = \int a \, dt + c$

If we plot relationships graphically we find that the area under a **velocity-time graph** represents the distance travelled by an object.

# Velocity and Acceleration

The area under an **acceleration time graph** represents the velocity of the object.

Areas under curves – a job for integration.

# Applying the Area Under a Curve - 1

- Example
  - The velocity of a body, v metres per second, after a time t seconds is given by $v = t^2+1$. Find the distance travelled at the end of 2 seconds.
  - When t = 0 the distance travelled will be 0 metres.
  - Hence the distance travelled at the end of 2 seconds is found by integrating the expression for v between the limits of 2 and 0.
  - This now gives us a means of expressing the information as an equation.

# Applying the Area Under a Curve - 2

Remember our equation for velocity given as $v = \frac{ds}{dt}$ can be represented in terms of distance (s) by integration

$$s = \int v \; dt + c$$

Our equation for velocity v = $t^2 + 1$ now becomes $s = \int_0^2 (t^2 + 1)dt$

Therefore we can perform integration using the limits of 0 and 2 to calculate distance travelled

$$\left[\frac{t^3}{3} + t\right]_0^2 = \frac{2^3}{3} + 2 = 4\tfrac{2}{3} \text{ metres}$$

# Acceleration - 1

Example

The acceleration of a moving body at the end of t seconds from the commencement of motion is (9 – t) metres per second.

Find the velocity and the distance travelled at the end of 2 seconds if the initial velocity is 5 metres per second.

As we know that the equation for acceleration is $v = \int a \ dt + c$

Then we can express acceleration as $\int (9 - t) dt + c$

# Acceleration - 2

By doing so it is then possible through integration to get the following equation for the velocity $v = \int 9t - \frac{t^2}{2} + c$

The initial velocity is the velocity when t = 0. However in order to calculate velocity we first need to calculate the constant of integration.

Hence when t = 0, v = 5

# Acceleration - 2

Therefore **5 = 9 x 0 – 0 + c** or put another way **c = 5**

Our equation for velocity is therefore $v = 9t - \frac{t^2}{2} + 5$

# Acceleration - 3

When t = 2, v = 9 x 2 - $\frac{2^2}{2}+5$ = **21 metres per second**

Given that distance s = $\int v\,dt$ = $\int (9t-\frac{t^2}{2}+5)dt+c$

Integrating gives $\frac{9t^2}{2}-\frac{t^3}{6}+5t+c$

Unless information is given to the contrary it is always assumed that s = 0 when t = 0 therefore c = 0

# Acceleration - 4

As distance $s = \dfrac{9t^2}{2} - \dfrac{t^3}{6} + 5t$

When t = 2, we can substitute into the equaton and calculate our distance

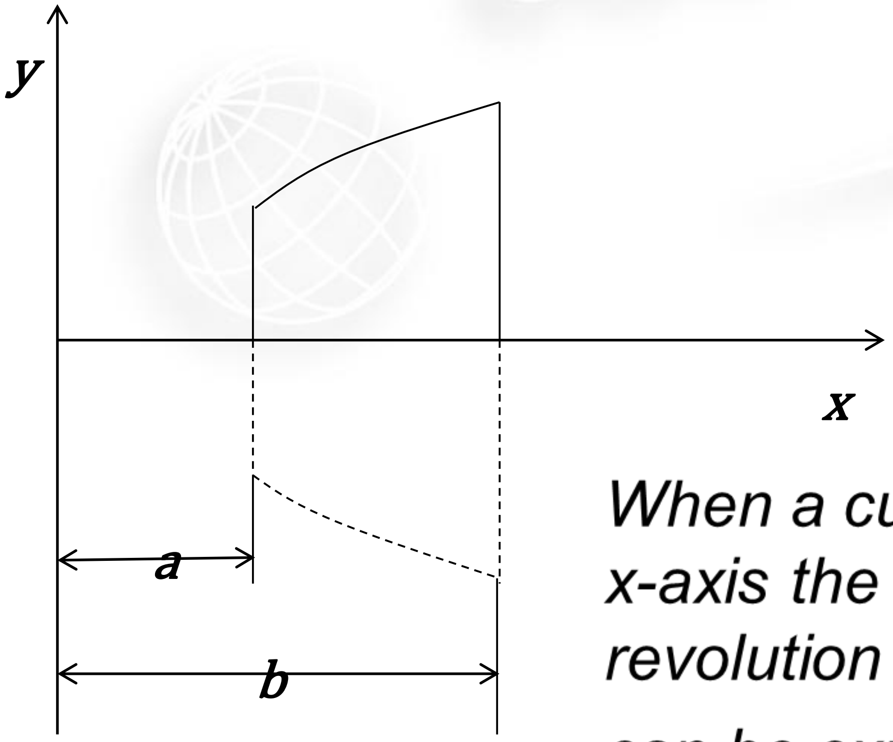
$s = \dfrac{9 \times 2^2}{2} - \dfrac{2^3}{6} + 5 \times 2 = 26\tfrac{2}{3}$ metres

# Solids of Revolution

- As we know from our previous experience of integration we can use this technique to calculate the area under a curve.
- If the area under a curve is rotated about the x-axis, the solid which results is called a solid of revolution. Any section of this solid by a plane perpendicular to the x-axis is a circle.

# Solids of Revolution

This diagram illustrates that if we rotate the shape formed by The area under the curve we get a solid of revolution




*When a curve is rotated about the x-axis the volume of the solid of revolution so produced can be expressed as* $\int_a^b \pi y^2 dx$

# Applying the Area Under a Curve - 1

- Once this relationship is established it is possible to calculate the volume of a range of solid objects that would otherwise be very difficult or impossible to do.
- Example
  - The area between the curve $y = x^2$, the x-axis and the ordinates at $x = 1$ and $x = 3$ is rotated about the x-axis.
  - Calculate the volume of the solid generated.

# Applying the Area Under a Curve - 2

As we are already in possession of the equation for calculating the volume of a solid of rotation and also have our upper and lower limits (x = 1 and x = 3) it is simply a process of integration using these limits, thus:

$$v = \int_1^3 \pi y^2 dx = \quad \int_1^3 (x^2)^2 dx = \quad \int_1^3 x^4 dx$$

We can now calculate the volume by applying the limits

$$\pi\left[\frac{x^5}{5}\right]_1^3 = \pi\left[\frac{243}{5} - \frac{1}{5}\right] = 48.4\pi$$ cubic units

# Topic 5 - Introduction to Integral Calculus 3

*Any Questions?*

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