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Foundation Mathematics

Topic 1 – Lecture 2: Introduction to Algebra

Factorisation

Algebraic Fractions

Scope and Coverage

This topic will cover:

- Factorisation of a range of algebraic expressions
- Algebraic fractions

Learning Outcomes

By the end of this topic students will be able to:

- Be able to perform a range of algebraic calculations
- Factorise algebraic expressions using a range of techniques
- Manipulate algebraic fractions

Terminology

- Terminology will be explained in the lecture, seminar and tutorial
- Ask questions if there is anything that you don’t understand

Factorisation – H.C.F. 1

- Highest Common Factor (H.C.F.)
- The H.C.F. is the highest expression which is a **factor** of each of the given expressions.
- To find the H.C.F. we select the lowest power of each of the quantities in all of the expressions and multiply them together.

Factorisation – H.C.F. 2

- Find the H.C.F. of ab^2c^2, $a^2b^3c^3$, $a^2b^4c^4$. Each expression contains the quantities *a*, *b* and *c*.
- To find the H.C.F. choose the lowest power of each of the quantities in the three expressions and multiply them together. The lowest power of *a* is *a*, the lowest power of *b* is b^2 and the lowest power of *c* is c^2.
- H.C.F. = ab^2c^2

Factorisation - Terms

- A factor is a common part of two or more **terms**. Thus the expression $3x+3y$ has two terms which have the number 3 common to both of them.
- Thus $3x+3y=3(x+y)$
- We say that 3 and $(x+y)$ are the factors of $3x+3y$

Factorisation – Finding Factors - 1

- To factorise these expressions, firstly we find the Highest Common Factor (H.C.F.) of all the terms making up the expression. The H.C.F. then appears outside the bracket. To find the terms inside the bracket divide each of the terms making up the expression by the H.C.F.

- Find the factors of $ax + bx$
- The HCF of ax and bx is x

$\therefore ax+bx=x(a+b)$ since $\dfrac{ax}{x}=a$ and $\dfrac{bx}{x}=b$

Factorisation – Finding Factors - 2

- Find the factors of $m^2n - 2mn^2$ the HCF of m^2n and $2mn^2$ is mn

$$\therefore m^2n - 2mn^2 = mn(m - 2n)$$

Since $\dfrac{m^2n}{mn} = m$ and $\dfrac{2mn^2}{mn} = 2n$

Factorisation – Finding Factors - 3

- Find the Factors of $\frac{ac}{x}+\frac{bc}{x^2}-\frac{cd}{x^3}$

The HCF of $\frac{ac}{x}$; $\frac{bc}{x^2}$ and $\frac{cd}{x^3}$ is $\frac{c}{x}$

$$\therefore \frac{ac}{x}+\frac{bc}{x^2}-\frac{cd}{x^3}=\frac{c}{x}\left(a+\frac{b}{x}-\frac{d}{x^2}\right)$$

Since $\frac{ac}{x}\div\frac{c}{x}=a$ $\frac{bc}{x^2}\div\frac{c}{x}=\frac{b}{x}$ and $\frac{cd}{x^3}\div\frac{c}{x}=\frac{d}{x^2}$

Factorisation – Multiple Terms - 1

- A **binomial expression** consists of two terms. Thus 3x+5, a+b, 2x+3y and 4p-q are all binomial expressions.
- If we want to consider the **product** of two binomial expressions we can do this in the same manner as if we were multiplying numbers hence the product of (a+b)(c+d) can be found by

$$\begin{array}{r} a+b \\ \times \quad c+d \\ \hline +da+db \\ ac+cb \\ \hline ac+bc+ad+bd \end{array}$$

Factorisation – Multiple Terms - 2

- If we consider (a+b)(c+d) = ac + ad + bc + bd
- The expression on the right hand side is obtained by multiplying each term in one bracket by each term in the other bracket
- Examples:

$(3x+2)(4x+5) = 3x \times 4x+3x \times 5+2 \times 4x+2 \times 5 = 12x^2+15x+8x+10 = 12x^2+23x+10$

$(2p-3)(4p+7)=2p \times 4p+2p \times 7-3 \times 4p-3 \times 7 = 8p^2+14p-12p-21 = 8p^2+2p-21$

$(z-5)(3z-2) = z \times 3z+z \times (-2)-5 \times 3z-5 \times (-2) = 3z^2-2z-15z+10 = 3z^2-17z+10$

Factorisation – Squares - 1

- The square of a binomial expression
- $(a+b)^2 = (a+b)(a+b) = a^2+ab+ba+b^2 = a^2+2ab+b^2$
- $(a-b)^2 = (a-b)(a-b) = a^2-ab-ba+b^2 = a^2-2ab+b^2$
- The square of a binomial expression is the sum of the squares of the two terms and twice their product.

Examples

$(2x+5)^2$	=	$(2x)^2+2\times 2x\times 5+5^2$	=	4x2+20x+25
$(3x-2)^2$	=	$(3x)^2+2\times 3x\times (-2)+(-2)^2$	=	$9x^2+12x+4$
$(2x+3y)^2$	=	$(2x)^2+2\times 2x\times 3y+(3y)^2$	=	$4x^2+12xy+9y^2$

Factorisation – Squares - 2

- The sum and difference of two terms
  - $(a+b)(a-b) = a^2-ab+ba-b^2 = a^2-b^2$
- This result is the difference of the squares of the two terms.

Examples

$(8x+3)(8x-3)= (8x)^2- 3^2 = 64x^2 - 9$

$(2x+5y)(2x-5y) = (2x)^2-(5y)^2 = 4x^2-25y^2$

- Where the factors form a perfect square
- If $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 -2ab + b^2$
- The square of a binomial expression therefore consists of: (square of 1st term) + (twice the product of the two terms) + (square of 2nd term).

Factorisation - Grouping

Factorising by grouping

- To factorise the expression ax+ay+bx+by the terms must be grouped in pairs so that each pair of terms has a common factor. For example:

  ax+ay+bx+by = (ax+ay)+(bx+by) = a(x+y)+b(x+y)

- In the two terms a(x+y) and b(x+y), the relationship (x+y) is a common factor. Therefore it is possible to write

  a(x+y)+b(x+y) = (x+y)(a+b)

  ∴ ax+ay+bx+by = (ax+by)+(bx+by)=a(x+y)+b(x+y)

Factorisation – Quadratic Expressions - 1

- A **quadratic expression** is one in which the highest power of the symbol used is the square. For instance, x^2-5x+3 and $3x^2-9$ are both quadratic expressions.
- Case 1. Where the coefficient of the squared term is unity.
  - $(x+4)(x+3) = x^2+7x+12$
- Note that in the quadratic expression $x^2+7x+12$ the last term 12 has the factors 4×3. Also the coefficient of x is 7 which is the sum of the factors 4 and 3.

Factorisation – Quadratic Expressions - 2

- Consider the following examples:
- Factorise x^2+6x+5. We note that 5 = 5 × 1 and 5+1 = 6 $\therefore x^2+6x+5 = (x+5)(x+1)$
- Factorise $x^2-8x+15$ Now 15 = 15 × 1 or 5 × 3 or (-5) × (-3) or (-1) × (-5)
- But the sum of the factors must be -8. Since -5+(-3) = -8, then $x^2-8x+15 = (x-5)(x-3)$

Factorisation

- In the following example $2x^2+5x-3$ is can be identified that the coefficient of x^2 is in fact preceded by a numerical value therefore the effect of this is to create a number of different possible factors for the expression thus:

  Factorising $2x^2+5x-3$

Factors of $2x^2$		Factors of -3	
2x	x	-3 +1	or +3 -1

- The combinations of these factors are:
  - $(2x-3)(x+1) = 2x^2-x-3$ which is incorrect
  - $(2x+1)(x-3) = 2x^2-5x-3$ which is incorrect
  - $(2x+3)(x-1) = 2x^2+x-3$ which is incorrect
  - $(2x-1)(x+3) = 2x^2+5x-3$ which is correct
- Hence $2x^2+5x-3 = (2x-1)(x+3)$

Fractions – Simplify 1

- As with ordinary arithmetic fractions, numerators can be multiplied together as can denominators, in order to form a single fraction, so:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

- Those factors which are common to both numerator and denominator may be cancelled out thus **simplifying** the expression

$$\frac{8ab}{3mn} \times \frac{9n^2m}{4ab^2} = \frac{\cancel{8} \times \cancel{a} \times \cancel{b} \times \cancel{9} \times \cancel{n} \times n \times \cancel{m}}{\underset{2}{\cancel{3}} \times \cancel{m} \times \cancel{n} \times \underset{3}{\cancel{4}} \times \cancel{a} \times \cancel{b} \times b} = \frac{6n}{b}$$

Fractions – Simplify 2

- Look to identify expressions that may be factorised Then you can cancel factors which are common to both numerator and denominator.
- (x-y), x(y+z) and (a-3)(a-5) may be regarded as single terms.
- Simplify:

$\frac{x^2 - x}{x - 1}$ As $x^2 - x = x(x-1)$ it follows that $\frac{x^2 - x}{x - 1} = \frac{x(x-1)}{(x-1)} = x$

Fractions – Simplify 3

Simplify $\dfrac{3xy-6x+y-2}{y^2-4}$

It should be remembered that simplification of expressions can help therefore looking at the top line of the expression we can derive

$$3xy-6x+y-2=(3xy-6x)+(y-2)=3x(y-2)+(y-2)=(y-2)(3x+1)$$

also $y^2-4=(y+2)(y-2)$

Therefore $\therefore \dfrac{3xy-6x+y-2}{y^2-4}=\dfrac{(3xy-6x)+(y-2)}{(y+2)(y-2)}=\dfrac{3x+1}{y+2}$

Fractions - Division

- To divide by a fraction **invert** it and then multiply:

Simplify

$$\frac{ax^2}{by} \div \frac{a^2}{b^2y^2} = \frac{ax^2}{by} \div \frac{a^2}{b^2y^2} = \frac{ax^2}{by} \times \frac{b^2y^2}{a^2} = \frac{bx^2y}{a}$$

$$\frac{3a^2+3am}{4a+6m} \div \frac{a^2+am}{4a+8m} = \frac{3a^2+3am}{4a+6m} \times \frac{4a+8m}{a^2+am} = \frac{3a(a+m)}{2(2a+3m)} \times \frac{4(a+2m)}{a(a+m)} = \frac{6(a+2m)}{2a+3m}$$

Fractions – Lowest Common Multiple (L.C.M.)

- To find the L.C.M. we select the highest power of each factor which occurs in any of the expressions.
- Examples - Find the L.C.M.
  - a^3b^2, abc^3, ab^3c The highest powers of a, b and c which occur in any of the given expressions are a^3, b^3 and c^3 $\therefore$ L.C.M. = $a^3b^3c^3$
  - $(x+4)^2$, $(x+4)(x+1)$. Since the contents of a bracket may be thought of as a single symbol. L.C.M. = $(x+4)^2(x+1)$
  - $(a+b)$, (a^2-b^2). Remember a^2-b^2 will factorise to give $(a+b)(a-b)$ $\therefore$ L.C.M. = $(a+b)(a-b)$

Fractions – Addition and Subtraction

- The method for algebraic fractions is the same as for arithmetical fractions, that is:

  1. Find the L.C.M. of the denominators.
  2. Express each fraction with the common denominators.
  3. Add or subtract the fractions.

- Example

  $\frac{2}{x}+\frac{3}{2x}-\frac{4}{3x}$ the LCM of $x,2x$ and $3x$ is $6x$

$$\frac{2}{x}+\frac{3}{2x}-\frac{4}{43x}=\frac{12+9+8}{6x}=\frac{29}{6x}$$

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Fractions – Further Examples

- Simplify

$$\frac{m}{12}+\frac{2m+n}{4}-\frac{m-2n}{3}$$

The L.C.M. of 12, 4 and 3 is 12

$$\therefore \frac{m}{12}+\frac{2m+n}{4}-\frac{m-2n}{3}=\frac{m+3(2m+n)-4(m-2n)}{12}=\frac{m+6m+3n-4m+8n}{12}$$

$$=\frac{3m+11n}{12}$$

Topic 1 – Introduction to Algebra 2

Any Questions?

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